UDC 531.36:534

STEADY OSCILLATIONS WITH POTENTIAL ENERGY HARMONIC IN SPACE AND PERIODIC IN TIME

PMM Vol. 43, No. 2, 1979, pp. 360 - 363

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(Received March 7, 1978)

It is shown that the answer to the question posed by A. Iu. Ishlinskii about whether the equilibrium position $x = 0$ of the system $x^{\cdot\cdot} = -\partial U(x, t)/\partial x$ can be stable, if function U is harmonic with respect to x, is positive. A symmetric time-periodic matrix $A(t)$ whose trace is zero, such that the zero solution of system $x^{\cdot\cdot} = A(t)x$ is stable, is constructed. (We recall that the quadratic form $(Ax, x)/2$ is harmonic then and only then, when the trace of A is zero). The configuration space dimension, i. e. the number of coordinates of vector x, is assumed greater than unity.

Although the proposed here example is somewhat artificial, the method of its construction may prove to be of some interest in the investigation of Lie semigroups and in the analysis of the attainable set boundary in controllable systems. This method is presented here in Sect. 3, where the origin of this example formally defined in Sect. 1, is explained. The equilibrium stability of the obtained system with harmonic potential is proved in Sect. 2.

1. Definition of the example. Let N and ε be positive numbers (below, N is selected fairly large and ε fairly small). We specify the symmetric matrix S of order n with zero trace so that the eigenvalues of matrix S^2 are positive and different. Such matrix exists for $n > 2$. (It is possible, for instance, to take a diagonal matrix that has the indicated properties). We shall assume that $n > 2$.

We subdivide segment $[0, (4/N) + \varepsilon)$ of the t-axis into six consecutive segments whose length Δ appears in the first row of the table below, and specify matrix $A(t)$ which in each of these segments is equal to the constant matrix shown in the second row of the table (the segment left-hand ends, but not the right-hand ones are assumed to be part of each segment).

Δ	$1/N$	$\varepsilon/3$	$1/N$	$1/N$	$2\varepsilon/3$	$1/N$
$A(t)$	$3NS/2$	0	$-3NS/2$	$3NS/4$	0	$-3NS/4$

Having thus constructed $A(t)$ on segment $[0, T)$, $T = (4/N) + \varepsilon$, we continue that function over the whole t-axis periodically with period T. Note that the trace of $A(t)$ is zero for all t, so that the potential which corresponds to $A(t)$ is harmonic.

Theorem 1. If we set ε fairly small and N fairly large, the equilibrium position $x = 0$ of the equation $x^{\cdot\cdot} = A(t)x$ with periodic coefficients is stable.

Note that in the example defined in Theorem 1 the dependence of function $A(t)$ on time is discontinuous. This defect can be easily eliminated by smoothing that function. Let us fix function A (i.e. ε and N in Theorem 1), and complement its curve by segments that link the limit values to the left and right at points of discontinuity. Let us consider the smooth periodic function A_* whose curve lies entirely in the δ-neighborhood of the supplemented curve of function A, and such that for any t the trace of $A_*(t)$ is zero. Such functions can be readily defined explicitly.

Theorem 2. If δ is fairly small, the equilibrium position $x = 0$ of the equation $x^{\cdot\cdot} = A_*(t)x$ with periodic coefficients is stable. Here $A_*(t) = -\partial U(x, t) / \partial x$ and the potential U is harmonic.

2. Proofs. Let us examine the monodromy operator of the equation $x^{\cdot\cdot} = A(t)x$ which is equivalent to the system $x^{\cdot} = y$, $y^{\cdot} = Ax$ with matrix

$$Q = \begin{Vmatrix} 0 & E \\ A & 0 \end{Vmatrix}$$

Hence the transformation of the phase space with matrix $\exp \Delta Q$ corresponds to a time segment of length Δ along which A is constant. The monodromy operator of the considered equation is the product of six of such exponents that correspond to six segments of constant A in a period.

First, let us consider segments of length $1 / N$ in each of which A is of the form NB, where $B = 3S / 2$, $-3S / 2$, $3S / 4$ and $-3S / 4$, respectively. Hence as $N \to \infty$ the related operator of the phase space transformation in the time $1 / N$ has the following finite limit:

$$\lim_{N\to\infty} \exp \frac{1}{N} \begin{Vmatrix} 0 & E \\ NB & 0 \end{Vmatrix} = \exp \begin{Vmatrix} 0 & 0 \\ B & 0 \end{Vmatrix} = \begin{Vmatrix} E & 0 \\ B & E \end{Vmatrix}$$

Consequently the product of the three exponents that correspond to the first three segments has the following finite limit as $N \to \infty$:

$$\exp \left[\begin{Vmatrix} E & 0 \\ -B & E \end{Vmatrix} \Delta \begin{Vmatrix} 0 & E \\ 0 & 0 \end{Vmatrix} \begin{Vmatrix} E & 0 \\ B & E \end{Vmatrix} \right], \quad B = \frac{3S}{2}, \quad \Delta = \frac{\varepsilon}{3}$$

Multiplication of the matrices yields

$$\exp \Delta \begin{Vmatrix} B & E \\ -B^2 & -B \end{Vmatrix}$$

The product of transformations of the last three segments is of the same form, except that in this case $\Delta = 2\varepsilon / 3$, and $B = 3S / 4$.

Finally, as $N \to \infty$ the matrix of the monodromy operator has the limit

$$M(\varepsilon) \doteq \exp \frac{2\varepsilon}{3} M_2 \exp \frac{\varepsilon}{3} M_1, \quad M_n = \begin{Vmatrix} 3S / 2^n & E \\ -9S^2 / 4^n & -3S/2^n \end{Vmatrix}, \quad n = 1, 2$$

When $\varepsilon \to 0$ we have

$$M(\varepsilon) = E + \varepsilon M_0 + \dots, \quad M_0 = \begin{Vmatrix} S & E \\ -9S^2 / 8 & -S \end{Vmatrix}$$

We shall prove that for fairly small ε all eigenvalues of $M(\varepsilon)$ are different with their modulus equal unity.

The formula for $M(\varepsilon)$ implies that for any fixed τ

$$\lim_{k\to\infty} M(\tau / k)^k = \exp \tau M_0 \qquad (2.1)$$

The transformation in the right-hand side has eigenvalues the modulus of all of which is equal unity, while being pairwise different (if τ is not too large). In fact, this is the transformation in time τ of the phase stream of the system

$$x^{\cdot} = Sx + y, \quad y^{\cdot} = -{}^9/_8 S^2 x - Sy$$

Eliminating y we obtain $x^{\cdot\cdot} = -{}^1/_8 S^2 x$, i. e. the equation of small oscillations with positive potential energy (the eigenvalues of S^2 are positive) and pairwise different natural frequences ω_j (the eigenvalues ω_j^2 of matrix S^2 are pairwise different). The eigenvalues of matrix $\exp \tau M_0$ are equal, hence $\exp(\pm i\tau\omega_j)$. These numbers are pairwise different for small τ.

Note that all of the considered here differential equations are Hamiltonian equations. Hence the operators of phase space transformation are simplicial. Simplicial operators whose eigenvalue moduli are all equal unity, are different from ± 1, and pairwise different, have the property of strong stability: they are stable together with all simplicial operators that are close to them (see [1]).

Thus the simplicial operator $\exp \tau M_0$ is highly stable. If follows now from formula (2.1) that when k is fairly large the operator $M(\tau / k)$ is highly stable, which means that for a fairly small ε the operator $M(\varepsilon)$ is highly stable. But then when N is fairly large but finite, the monodromy operator $M(N, \varepsilon)$ close to $M(\varepsilon)$ is also stable.

This proves Theorem 1. Theorem 2 follows from that the operator $M(N, \varepsilon)$ is highly stable, and the monodromy operator for A_* is fairly close to it.

Remarks. 1°. The above reasoning provides also an example of the one-dimensional system $x^{\cdot\cdot} = A(t)x$ with periodic coefficients, which remains stable when the sign of the right-hand side is changed. Moreover, the function with such properties can be selected odd (or even) with respect to t. Formulas in Sect. 1 in which $S = 1$ provide such an example with a discontinuous function A; the same properties can be obtained for a smooth function A by the smoothing procedure.

2°. It is now possible to construct an example of stable equation with a potential that is harmonic in space and time periodic in the case of $n = 2$. For this it is necessary to consider a system which decomposes into the direct product $x_1^{\cdot\cdot} = A(t)x_1$, $x_2^{\cdot\cdot} = -A(t)x_2$, where function A is defined in Remark 1°.

3. Lie semigroups and the attainability limit. 1°. This problem can be considered to be a particular case of the problem of Lie semigroups generated by the convex cones in Lie algebras. The Lie semigroup generated by a cone represents the closure of products of exponents of Lie algebra elements that lie in the cone.

In the considered problem the Lie group is a group of simplicial matrices of order $2n$, and the cone directrix consists of matrices of the form

$$\begin{Vmatrix} 0 & E \\ S & 0 \end{Vmatrix}$$

where S is a symmetric matrix whose trace is zero.

It was shown above that for $n \geqslant 3$ the semigroup generated by such cone contains only highly stable simplicial matrices. It would be interesting to try to classify the semigroups generated by convex cones belonging to classical Lie groups. The reasoning presented below makes possible the first steps in that direction. The example in Sect. 1 has been formulated on that basis.

2°. The problem of Lie semigroups is in turn a particular case of the problem of determination of the attainable set in controlled systems. The latter problem is generally formulated as follows: in the space tangent to a manifold a subset (called the indicatrix of possible velocities) is specified at every point. It is required to determine the set of points that can be reached from a given point in motion whose velocity at every instant of time belongs to the indicatrix, or to determine at least the closure of such set.

One of the ways of solving this problem consists of successive extension of the indicatrix, which is carried out as follows. First, instead of the indicatrix we can consider the cone drawn over it, since only the direction of the curve reaching it is essential, not the velocity of motion along it. Second, the obtained cone can be closed, hoping that points attainable with the new cone will lie either inside or at the boundary of the set of points attainable prior to the closure. Third, the obtained closed cone can be extended to its convex shell, considering the possible mixed strategies (i. e. of motions with rapidly changing sections along which the velocity is selected differently).

Let us assume that as the result of these operations we obtain such cones in the tangent spaces such that each of them contains in the tengent space some subspace. A fourth possibility of extending the cone-indicatrix exists in that case. Let us consider any vector field whose vector at each point lies in the considered subspace of the tangent space.

Let us consider some point on the phase curve passing through the selected reference point. Transformation invers to the transformation of the phase stream, which transforms the reference point to the end one, maps the end point indicatrix into the tangent space at the reference point. The shifted indicatrix can be joined to the reference point indicatrix.

It is actually possible to move for a certain time N over the vector field to the end point, then for some short time to follow the indicatrix of the end point, and then during time N move back along the phase curve of the vector field. As the result we shift from the reference point by a distance of order ε by following the shifted indicatrix (within smalls of higher order with respect to ε).

The combination of all indicatrices shifted in this manner can be again subjected to convexification, closure, etc., until these operations no longer alter the indicatrices.

3°. The example in Sect. 1 was derived as follows. The initial indicatrix in the unit of a simplicial group consists of matrices of the form

$$\begin{Vmatrix} 0 & E \\ A & 0 \end{Vmatrix}$$

where A is any symmetric matrix with zero trace. At the remaining points of the group the indicatrix is obtained from the latter by a right-hand shift.

Thus at each point the indicatrix is an affine plane, and the corresponding cone is an open half-plane of the number of dimensions greater by one. Its closure represents a closed half-plane, and the boundary (at point unity) consists of matrices of the form

$$\begin{Vmatrix} 0 & 0 \\ A & 0 \end{Vmatrix}$$

where A is a symmetric matrix with zero trace.

In this way we have obtained a right-invariant field of planes lying in the extended indicatrices. The multiplication of the group elements (from the right) by

$$\exp t \begin{Vmatrix} 0 & 0 \\ A & 0 \end{Vmatrix}$$

specifies the phase stream, whose velocity vectors belong to the derived planes. Using the extension of the indicatrix by shifts along that field we join to the indicatrix in the unity all matrices of the form

$$\begin{Vmatrix} E & 0 \\ A & E \end{Vmatrix} \begin{Vmatrix} 0 & E \\ B & 0 \end{Vmatrix} \begin{Vmatrix} E & 0 \\ -A & E \end{Vmatrix} = \begin{Vmatrix} A & E \\ B - A^2 & -A \end{Vmatrix}$$

where A and B are symmetric matrices with zero trace, and the first matrix in the left-hand side is one of matrices of phase stream transformation, and the last one is its inverse matrix.

We convexificate the set of all indicatrices shifted in this manner, and obtain the set of all matrices of the form

$$\begin{Vmatrix} S & E \\ C & -S \end{Vmatrix}$$

where S and C are symmetric matrices with the trace of S equal zero and $\mathrm{tr}C \leqslant - \mathrm{tr}\, S^2$. Among these matrices there are matrices of the form

$$\begin{Vmatrix} S & E \\ -\lambda S^2 & -S \end{Vmatrix}, \quad \lambda > 1$$

to which corresponds the system $x^{\cdot} = Sx + y,\ y^{\cdot} = -\lambda S^2 x - Sy$ or the equation $x^{\cdot\cdot} = -(\lambda - 1)S^2 x$ which defines stable oscillations. It is now not difficult to plot the path that leads from unity to a highly stable simplicial matrix, as was done in Sect. 1.

REFERENCES

1. Arnol'd, V. I., Mathematical Methods of Classical Mechanics. Moscow, "Nauka", 1974.

Translated by J. J. D.